ON THE VIBRATIONS OF A HOMOGENEOUS ELASTIC HALF-SPACE UNDER THE ACTION OF A SOURCE APPLIED TO A UNIFORMLY EXPANDING CIRCULAR REGION

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The study is concerned with the propagation of elastic waves produced by a moving source. This topic has been the subject of a series of papers, e. g. [1 to 3]. In contradistinction to the above cited works which study the effects of a moving point source, the present investigation deals with the effects of a source applied to a uniformly expanding circular region on the boundary of an elastic half-space. The resultant displacement field is divided into components corresponding to longitudinal, transverse, head, Rayleigh and other waves. Each of these waves is investigated in the prefrontal regions, and the connection between the character of the singularity and the relationship between the velocity of propagation and the velocity of the source is explained. The results obtained for the field of a moving source are compared with the cases of point and distributed stationary sources.

In contrast with [1 to 3], where the velocity of source motion is assumed to be lower than the velocity of propagation of Rayleigh waves, herein no limitations are placed on the velocity of motion. In studying the field, particular attention is paid to the cases in which the velocity of the source coincides with the velocities of propagation of longitudinal, transverse and Rayleigh waves. If the velocity of motion is equal to the Rayleigh velocity, resonance of surface waves takes place. Thereupon, the Rayleigh wave changes form and propagates without attenuation. If the velocity of motion coincides with either the longitudinal or transverse wave propation velocity, the form and attenuation of the waves retain the same character as in the case of a stationary source.

1. Consider a cylindrical coordinate system $r\theta z$ in a homogeneous elastic half-space $z \geqslant 0$, characterized by density ρ and Lamé constants λ and μ. The region is at rest at $t < 0$, and subjected from the time $t = 0$ to the action of a source applied to a circular region which is uniformly expanding with time and for which $r = vt$ and $z = 0$. The loading due to this source is given by the relations

$$t_{zz} = -\frac{\delta(r - vt)}{r}\,\varepsilon(t), \qquad t_{zr} = t_{z\theta} = 0 \quad \text{for} \quad z = 0 \tag{1.1}$$

Here t_{zz}, $t_{z\theta}$ and t_{zr} are components of the stress tensor, v is the velocity of the motion of the source, and $\delta(x)$ and $\varepsilon(x)$ are the Dirac and Heaviside functions, respectively.

The resultant displacement field $\mathbf{u} = q\mathbf{r}_1 + w\mathbf{k}_1$ is defined by the known Lamé Eqs.

$$\rho\partial^2\mathbf{u} / \partial t^2 = (\lambda + 2\mu) \text{ grad div } \mathbf{u} - \mu \text{ rot rot } \mathbf{u} \tag{1.2}$$

with the initial conditions

$$\mathbf{u} = \mathbf{u}^{\cdot} = 0 \quad \text{for } t = 0 \tag{1.3}$$

The boundary conditions are given by

$$\left(\frac{\partial q}{\partial z} + \frac{\partial w}{\partial r}\right) = 0, \left[\lambda\left(\frac{\partial q}{\partial r} + \frac{q}{r}\right) + (\lambda + 2\mu)\frac{\partial w}{\partial z}\right] = -\frac{\delta(r - vt)}{r}\varepsilon(t) \qquad \text{for } z = 0 \tag{1.4}$$

The solution of the problem formulated in (1.2) to (1.4) may be obtained, as in [4 and 5], by means of Fourier-Bessel and Laplace transformations. We omit intermediaate formulas, and write merely the end results

$$w = \int_0^\infty \frac{J_0(kr)\,dk}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} W(k, \eta) \exp\frac{kt\eta}{b}\,d\eta$$
$$q = \int_0^\infty \frac{J_1(kr)\,dk}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} Q(k, \eta) \exp\frac{kt\eta}{b}\,d\eta \tag{1.5}$$

$$W = \frac{\alpha}{\mu\varkappa R}(ge^{-kz\alpha} - 2e^{-kz\beta}), \qquad Q = \frac{1}{\mu\varkappa R}(ge^{-kz\alpha} - 2\alpha\beta e^{-kz\beta}) \tag{1.6}$$

$$\alpha = \sqrt{1 + \gamma^2\eta^2}, \qquad \beta = \sqrt{1 + \eta^2}, \qquad \varkappa = \sqrt{\eta^2 + b^2v^2}, \qquad g = 2 + \eta^2$$

$$R = g^2 - 4\alpha\beta, \qquad a = \sqrt{\rho(\lambda + 2\mu)^{-1}}, \qquad b = \sqrt{\rho\mu^{-1}}, \qquad \gamma = ab^{-1} \tag{1.7}$$

which define the displacement vector $\mathbf{u}$. In order that the radicals $\varkappa$, β and α be single-valued, branch cuts originating at the branch points $\pm ibv$, $\pm i$ and $\pm i\gamma^{-1}$ are introduced in the left half-plane, and the values of the radicals are fixed by the conditions $\varkappa > 0$, $\beta > 0$ and $\alpha > 0$ for $\eta > 0$.

Before proceeding with the investigation of the solutions (1.5) to (1.7), let us obtain the displacement field in the same half-space for two other types of sources

$$t_{zz} = -\frac{\delta(r)'}{r}\varepsilon(t), \qquad t_{zr} = t_{z\theta} = 0 \qquad \text{for } z = 0 \tag{1.8}$$

$$t_{zz} = -\frac{\delta(t)}{r}, \qquad t_{zr} = t_{z\theta} = 0 \qquad \text{for } z = 0 \tag{1.9}$$

It is readily seen that the first of the above sources represent that particular case of (1.1) for which the velocity of motion is zero. Hence, the displacement field produced by the source (1.8) is given by Formulas (1.5) wherein

$$W = \frac{\alpha}{\mu\eta R}(ge^{-kz\alpha} - 2e^{-kz\beta}), \qquad Q = \frac{1}{\mu\eta R}(ge^{-kz\alpha} - 2\alpha\beta e^{-kz\beta}) \tag{1.10}$$

To obtain the displacements resulting from the action of (1.9) we need merely note the relation

$$\frac{\delta(t)}{r} = \lim_{v \to \infty}\left[\frac{\delta(r - vt)\,v\varepsilon(t)}{r}\right] \tag{1.11}$$

between the functions defining the sources in (1.1) and (1.9). Utilizing (1.11), we find that the desired displacements are also expressible by (1.5) wherein

$$W = \frac{\alpha}{\mu bR}(ge^{-kz\alpha} - 2e^{-kz\beta}), \qquad Q = \frac{1}{\mu bR}(ge^{-kz\alpha} - 2\alpha\beta e^{-kz\beta}) \tag{1.12}$$

The displacement field (1.5) and (1.10) for a stationary point source (1.8) was examined in detail in [4], using exact and asymptotic methods. Similar methods may be used to study the displacement fields (1. 5), (1. 6) and (1. 12) for a moving source (1. 1) and a stationary distributed source (1. 9). Since the field of a moving source is of greatest interest, our attention will be mainly directed to its investigation. In addition to examining the influence of the source motion on the propagation of longitudinal, transverse, head and Rayleigh waves in the half-space, we will also study waves which propagate with the velocity of the source.

2. To investigate the displacement fields (1. 5) and (1. 6) let W and Q be written as

$$W = W_p e^{-kz\alpha} + W_s e^{-kz\beta}, \qquad Q = Q_p e^{-kz\alpha} + Q_s e^{-kz\beta} \tag{2.1}$$

where, from (1. 6),

$$W_p = \frac{\alpha g}{\mu R \varkappa}, \quad W_s = -\frac{2\alpha}{\mu R \varkappa}, \quad Q_p = \frac{g}{\mu R \varkappa}, \quad Q_s = -\frac{2\alpha\beta}{\mu R \varkappa} \tag{2.2}$$

Corresponding to the decomposition of functions in (2. 1) into two components, we will write the field $\mathbf{u}$ as the sum

$$\mathbf{u} = \mathbf{u}_p + \mathbf{u}_s \tag{2.3}$$

Here the components q_l and w_l of the vector $\mathbf{u}_l$ $(l = p,\ s)$ are given by

$$w_l = \int_0^\infty \frac{J_0(kr)\,dk}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} W_l \exp\left[k f_l(\eta)\right] d\eta \tag{2.4}$$

$$q_l = \int_0^\infty \frac{J_1(kr)\,dk}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} Q_l \exp\left[k f_l(\eta)\right] d\eta$$

$$f_p(\eta) = t\eta / b - z\alpha, \qquad f_s(\eta) = t\eta / b - z\beta \tag{2.5}$$

The quantities w_l and q_l will be studied by asymptotic methods. The inside, Mellin integrals will be evaluated by the method of steepest descent in the region $t > az$ $(l = p)$ or $t > bz$ $(l = s)$, where the integrals are not identically zero.

In this connection, let us examine the stationary contours λ_p and λ_s of the phases of functions (2. 5). The contours λ_p and λ_s pass through the saddle points

$$\pm \eta_{p0} = \frac{\pm it}{\gamma \sqrt{t^2 - a^2 z^2}}, \qquad \pm \eta_{s0} = \frac{\pm it}{\sqrt{t^2 - b^2 z^2}} \tag{2.6}$$

satisfying the conditions

$$\operatorname{Im} f_l(\eta) = \operatorname{Im} f_l(\pm \eta_{l0}), \qquad \operatorname{Re} f_l(\eta) \leqslant \operatorname{Re} f_l(\pm \eta_{l0}) \tag{2.7}$$

intersecting the imaginary axis at the points

$$\pm \eta_{p1} = \frac{\pm i \sqrt{t^2 - a^2 z^2}}{\gamma t}, \qquad \pm \eta_{s1} = \frac{\pm i \sqrt{t^2 - b^2 z^2}}{t} \tag{2.8}$$

and possessing symmetry with respect to the real axis. Possible arrangements of stationary contours in the upper half-plane are shown in Figs. 1 and 2.

In order to go from the Mellin contour in (2. 4) to the stationary contour λ_l it is necessary to examine and take into account the singularities of the integrand in the region between the two contours. In that region, there can only be the branch points $\pm i, \pm i\gamma^{-1}, \pm ibv$ and the poles $+ i\tau_R$, satisfying the Rayleigh equation $R = 0$. It is easily seen that the noted singular points fail to fall in the region between the two contours only if they are located on the imaginary axis between the points η_{l0} and η_{l1}. Thus, in passing from the Mellin contour to the stationary contour it is necessary to take into account singularities at the point $\pm i$, $\pm i\gamma^{-1}$, $\pm ibv$ and $\pm i\tau_R$ if these lie outside the interval $(\pm \eta_{l1}, \pm \eta_{l0})$ on the imaginary axis. These singularities are taken into account by evaluating the residues and the integrals along the branch cuts. In order to utilize the method of steepest descent in evaluating the integrals, we introduce branch cuts from the points $\pm i$, $\pm i\gamma^{-1}$ and $\pm ibv$, along the lines

$$\operatorname{Im} f_l(\eta) = \operatorname{Im} f_l(\pm i), \quad \operatorname{Im} f_l(\eta) = \operatorname{Im} f_l(\pm i\gamma^{-1}), \quad \operatorname{Im} f_l(\eta) = \operatorname{Im} f_l(\pm ibv) \tag{2.9}$$

in the left half of the η-plane (Figs. 1 and 2). For convenience, the contours encompassing these cuts will be designated by λ_{l1}, $\lambda_{l\gamma}$ and λ_{lv} respectively. Based on these investigations, the displacement fields (2.4) and (2.2) may be represented by

$$\mathbf{u}_p = \mathbf{u}_{pv} + \mathbf{u}_{p0} + \mathbf{u}_{p\gamma} + \mathbf{u}_{pR}, \quad \mathbf{u}_s = \mathbf{u}_{sv} + \mathbf{u}_{s0} + \mathbf{u}_{s1} + \mathbf{u}_{sR} \tag{2.10}$$

where the components of the vector $\mathbf{u}_{lv}$, $\mathbf{u}_{l0}$, $\mathbf{u}_{p\gamma}$, $\mathbf{u}_{s1}$ and $\mathbf{u}_{lR}$ may be expressed in terms of Fourier-Bessel integrals obtained from the integrals along the respective contours λ_{lv}, λ_l, $\lambda_{p\gamma}$, λ_{s1} and from the residues at the point $\pm i\tau_R$. Here we should note that the vectors $\mathbf{u}_{lv}$, $\mathbf{u}_{p\gamma}$, $\mathbf{u}_{s1}$ and $\mathbf{u}_{lR}$ vanish if the corresponding singularities $\pm ibv$, $\pm i\gamma^{-1}$, $\pm i$ and $\pm i\tau_R$ lie inside the intervals $(+\eta_{l1}, \pm\eta_{l0})$ on the imaginary axis.

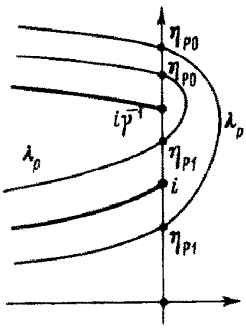

Fig. 1.

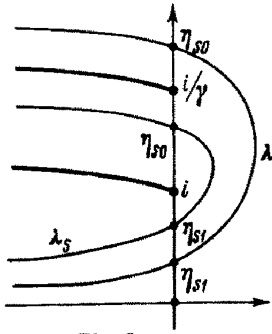

Fig. 2.

3. In order to investigate the singularities of the vector fields $\mathbf{u}_{l0}$, $\mathbf{u}_{p0}$, $\mathbf{u}_{p\gamma}$ and $\mathbf{u}_{s1}$ let us divide the range of integration with respect to k into two intervals: $[0, k_0]$ and $[k_0, \infty)$. Since the integrands in the integrals along the contours λ_{lv}, λ_p, $\lambda_{s\gamma}$ and λ_{1p} are regular functions over the interval $[0, k_0]$, integration over the finite interval $[0, k_0]$ is of no interest. All singularities of the vectors $\mathbf{u}_{lv}$, $\mathbf{u}_{l0}$, $\mathbf{u}_{p\gamma}$ and $\mathbf{u}_{s1}$ will be contained in the integrals over the range $[k_0, \infty)$. The value of k_0 must be chosen that

$$k_0 r \gg 1, \qquad k_0 t b^{-1} \gg 1 \tag{3.1}$$

Under these conditions, the Bessel functions $J_0(kr)$ and $J_1(kr)$ may be replaced by their asympotic forms

$$J_n(kr) \sim \sqrt{2/\pi kr}\, \cos(kr - {}^1/_2 n\pi - {}^1/_4\pi) \tag{3.2}$$

and the integrals along λ_{lv}, λ_{l0}, $\lambda_{\gamma s}$ and λ_{p1} may be evaluated by the method of steepest descent. Application of these methods to the evaluation of expressions typified by $\mathbf{u}_{lv}$, $\mathbf{u}_{l0}$, $\mathbf{u}_{s\gamma}$ and $\mathbf{u}_{p1}$ is described in detail in [4 and 5]. Hence, we will confine ourselves here to obtaining the leading asympotic terms in the nonanalytic portion of the field in the neighbourhood of the front.

The $\mathbf{u}_{pv}$ wave. The components q_{pv}, w_{pv} of the vector $\mathbf{u}_{pv}$ under conditions $|\eta_{p0}| < bv$ or $z < t\sqrt{v^2 - a^{-2}}$ are given by

$$q_{pv} = \frac{g(ibv)}{\pi\mu R(ibv)\sqrt{bvr f_p'(ibv)}} \int_{k_0}^{\infty} \frac{\sin k(r - vt + z\sqrt{a^2v^2 - 1})}{k}\, dk$$

$$w_{pv} = \sqrt{a^2v^2 - 1}\; q_{pv} \tag{3.3}$$

and have singularities only on the surface

$$vt = z\sqrt{a^2v^2 - 1} + r \tag{3.4}$$

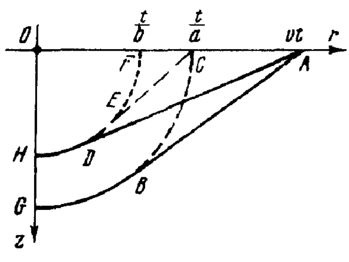

Fig. 3.

This conical surface makes an angle $\alpha_1 = \sin^{-1}(av)^{-1}$ with the plane $z = 0$. The trace of the surface is represented in Fig. 3 by the segment AB. In passing through the surface (3.4), the displacement components (3.3) experience jumps, so that the integral

$$\int_{k_0}^{\infty} \sin kx \frac{dk}{k}$$

has a jump equal to π at $x = 0$.

If one of the relations $|\eta_{p1}| > bv$ or $az < t\sqrt{1 - a^2v^2}$ is satisfied, then both components q_{pv} and w_{pv} are represented by integrals of the type

$$\int_{k_0}^{\infty} \exp\,[ik(r - vt) - kz\sqrt{1 - a^2v^2}]\,\frac{dk}{k} \tag{3.5}$$

which differ from the integrals in (3.3) by a factor $\exp(-kz\sqrt{1 - a^2v^2})$. As a result of this factor, the wave represented by the vector $\mathbf{u}_{pv}$, is attenuated exponentially with an increase in z, and thus is a surface wave. The displacement field of this wave has singularities only on the surface $z = 0$.

The $\mathbf{u}_{sv}$ wave. Investigation of the displacement field $\mathbf{u}_{sv}$ differs little from the preceding studies. Under conditions $|\eta_{s1}| < bv$ or $bz < t\sqrt{b^2v^2 - 1}$ the components q_{sv} and w_{sv} are given by Expressions

$$w_{sv} = -A_{sv}\left[\operatorname{Im}\frac{\alpha(ibv)}{R(ibv)}\int_{k_0}^{\infty}\sin k\omega_{sv}\,\frac{dk}{k} + \operatorname{Re}\frac{\alpha(ibv)}{R(ibv)}\int_{k_0}^{\infty}\cos k\omega_{sv}\,\frac{dk}{k}\right]$$

$$q_{sv} = -\sqrt{b^2v^2 - 1}\,w_{sv} \tag{3.6}$$

$$A_{sv} = [\pi\mu\sqrt{bvrf_s'(ibv)}]^{-1}, \qquad \omega_{sv} = r - vt + z\sqrt{b^2v^2 - 1}$$

On the conical surface AD (Figs. 3 and 4)

$$vt = r + z\sqrt{b^2v^2 - 1} \tag{3.7}$$

which makes an angle $\beta_1 = \sin(bv)^{-1}$ with the surface $z = 0$ the functions (3.6) posses a singularity.

The character of this singularity is determined by the relationship of the source velocity v to the velocity of propagation a^{-1}. If $v > a^{-1}$, then $\operatorname{Re}[\alpha(ibv)R^{-1}(ibv)] = 0$, and the functions (3.6) execute jumps AD (Fig. 3) on the surface (3.7).

In the case $a^{-1} < v < b^{-1}$ the quantities $\alpha(ibv)R^{-1}(ibv)$ will be complex, and the components in (3.6) will have jumps as well as logarithmic singularities AD (Fig. 4) on the front (3.7). In the case $bz < t\sqrt{1 - b^2v^2}$, corresponding to the condition $|\eta_{s1}| > bv$, the $\mathbf{u}_{sv}$ wave is a surface wave, exponentially decaying with depth.

Longitudinal wave $\mathbf{u}_{p0}$. The components q_{p0} and w_{p0} of the vector $\mathbf{u}_{p0}$ are given by the relations for $bv < |\eta_{p0}|$

$$q_{p0} = \frac{g(\eta_{p0}) A_{p0}}{\sqrt{|\eta_{p0}|^2 - b^2v^2}} \int_{K_0}^{\infty} \frac{\sin k\omega_{p0}}{k} dk$$

$$w_{p0} = |\alpha(\eta_{p0})| q_{p0} \tag{3.8}$$

for $bv > |\eta_{p0}|$

$$q_{p0} = -\frac{g(\eta_{p0}) A_{p0}}{\sqrt{b^2v^2 - |\eta_{p0}|^2}} \int_{K_0}^{\infty} \frac{\cos k\omega_{p0}}{k} dk$$

$$w_{p0} = |\alpha(\eta_{p0})| q_{p0} \tag{3.9}$$

The following expressions have been introduced in (3. 8) and (3. 9):

$$A_{p0} = [\pi\mu R(\eta_{p0}) \sqrt{r|f_p''(\eta_{p0})|}]^{-1}, \quad \omega_{p0} = r - a^{-1}\sqrt{t^2 - a^2z^2} \tag{3.10}$$

It is easily seen that the expressions in (3. 8) and (3. 9) become singular only on the sphere

$$r^2 + z^2 = a^{-2}t^2 \tag{3.11}$$

In order to examine the properties of the field in the neighborhood of the sphere (3. 11), we introduce the angle $\varphi = \sin^{-1}(art^{-1}) = \tan^{-1}(rz^{-1})$, at which the ray originating at the point (r, z) of the sphere is propagated. In view of relations (3. 8) and (3. 9), the field of the vector $\mathbf{u}_{p0}$ possesses on the front (3, 11) BG(Fig. 3) and CG(Fig 4), jump type singularities, when $\sin\varphi < (av)^{-1}$ and BC(Fig. 3), logarithmic singularities, when $\sin\varphi > (av)^{-1}$

T r a n s v e r s e w a v e $\mathbf{u}_{s0}$. The approximate relations for the components q_{s0} and w_{s0} of vector $\mathbf{u}_{s0}$ are given by

$$w_{s0} = -A_{s0}\left[\mathrm{Re}B_{s0} \int_{K_0}^{\infty} \frac{\sin k\omega_{s0} dk}{k} - \mathrm{Im}\, B_{s0} \int_{K_0}^{\infty} \frac{\cos k\omega_{s0} dk}{k}\right], \tag{3.12}$$

$$q_{s0} = -|\beta(\eta_{s0})| w_{s0}, \qquad \omega_{s0} = r - b^{-1}\sqrt{t^2 - b^2z^2}$$

$$A_{s0} = 2[\pi\mu \sqrt{r|f_s''(\eta_{s0})|}]^{-1}, \qquad B_{s0} = \alpha(\eta_{s0})[R(\eta_{s0}) \varkappa(\eta_{s0})]^{-1} \tag{3.13}$$

The right-hand sides of (3. 12) possess singularities on the sphere

$$r^2 + z^2 = b^{-2}t^2 \tag{3.14}$$

The trace of the sphere (3. 14) in Figs. 3 to 5 is given by the arc FH. To find the character of the field in the neighborhood of the sphere (3. 14), it is convenient to introduce the angle $\varphi = \sin^{-1}(brt^{-1}) = \tan^{-1}(rz^{-1})$, at which the ray originating at the point (r, z) of the sphere propagates. The singularity of the vector $\mathbf{u}_{s0}$ on the sphere is determined by the angle φ, which is related to the saddle point η_{s0} by Eq. $\sin\psi = |\eta_{s0}^{-1}$. In case $\sin\psi < \gamma$ and $\sin\psi < (bv)^{-1}$(Fig. 3, DH; Fig. 4, EH), the value of B_{s0} is real and the functions (3. 12) have jumps on the sphere (3. 14).
If $\gamma > \sin\psi > (bv)^{-1}$ (Fig. 4, DE), B_{s0} is imaginary and the singularity at the front is logarithmic. Finally, for $\sin\psi > \gamma$ (Fig. 3 to 5, EF), B_{s0} is complex and the function in (3. 12) have both jumps and logarithmic singularities at the front.

Head wave $\mathbf{u}_{s\gamma}$. The field of the displacement vector $\mathbf{u}_{s\gamma}$ may be represented as

$$w_{s\gamma} = \frac{A_\gamma}{\sqrt{1-a^2v^2}} \int_{k_0}^{\infty} \cos k\left(r - \frac{t}{a}\,\frac{z\sqrt{1-\gamma^2}}{\gamma}\right)\frac{dk}{k^2} \quad (v < a^{-1}) \tag{3.15}$$

$$w_{s\gamma} = \frac{A_\gamma}{\sqrt{a^2v^2-1}} \int_{k_0}^{\infty} \sin k\left(r - \frac{t}{a} - \frac{z\sqrt{1-\gamma^2}}{\gamma}\right)\frac{dk}{k^2} \quad (v > a^{-1}) \tag{3.16}$$

$$A_\gamma = 2\gamma\,[\pi\mu g^2(i\gamma^{-1})\,f_s'(i\gamma^{-1})\sqrt{rf_s'(i\gamma^{-1})}]^{-1}, \qquad q_{s\gamma} = -\gamma^{-1}\sqrt{1-\gamma^2}\,w_{s\gamma} \tag{3.17}$$

On the basis of relations (3.15) and (3.16) it is easily seen that the vector field $u_{s\gamma}$ (t,r,z) is continuous and that the first derivatives of the vector $\mathbf{u}_{s\gamma}$ (for example, the velocity $\mathbf{u}_{s\gamma}$) have singularities on the conical surface

$$ar + bz\sqrt{1-\gamma^2} = t \tag{3.18}$$

which make an angle $\gamma_1 = \sin^{-1}\gamma$ with the z-axis. On the surface (3.18), the vector $\mathbf{u}_{s\gamma}$ has a jump (Fig. 4, 5, CE) for $v < a^{-1}$ and logarithmic singularity (Fig. 3, CE) for $v > a^{-1}$.

The $\mathbf{u}_{p1}$ wave is represented by integrals of the type

$$\int_{k_0}^{\infty} \frac{\exp\,[\pm ik\,(tb^{-1} - r) - kz\sqrt{1-\gamma^2}]\,dk}{k^2} \tag{3.19}$$

This wave, just as the $\mathbf{u}_{pv}$ wave for $v < a^{-1}$, is a surface wave. The singularities of this wave exist only on the circle $br = t$, $z = 0$ (Fig. 3 to 5, F). Since the $\mathbf{u}_{s0}$ wave has a stronger singularity on this circle, the $\mathbf{u}_{p1}$ wave is of no interest.

The Rayleigh wave $\mathbf{u}_R = \mathbf{u}_{pR} + \mathbf{u}_{sR}$, is composed of longitudinal $(\mathbf{u}_{pR})$ and transverse $(\mathbf{u}_{sR})$ parts, and, unlike the other waves, it may be evaluated exactly. Omitting intermediate expressions, the relations for the components g_R and w_R of $\mathbf{u}_R$ are:

$$q_R = \frac{A_R}{r\sqrt{v_R^2 - v^2}}\left[g_R\left(1 - \frac{\alpha_R z\cos\psi_\gamma + v_R t\sin\psi_\gamma}{T_\gamma^{1/4}}\right) - 2\alpha_R\beta_R\left(1 - \frac{\beta_R z\cos\psi_1 + v_R t\sin\psi_1}{T_1^{1/4}}\right)\right]$$

$$w_R = \frac{\alpha_R A_R}{\sqrt{v_R^2 - v^2}}\left[g_R\frac{\cos\psi_\gamma}{T_\gamma^{1/4}} - 2\frac{\cos\psi_1}{T_1^{1/4}}\right] \quad (v < v_R) \tag{3.20}$$

$$q_R = -\frac{A_R}{r\sqrt{v^2 - v_R^2}}\left[g_R\left(1 - \frac{\alpha_R z\sin\psi_\gamma - v_R t\cos\psi_\gamma}{T_\gamma^{1/4}}\right) - 2\alpha_R\beta_R\left(1 - \frac{\beta_R z\sin\psi_1 - v_R t\cos\psi_1}{T_1^{1/4}}\right)\right]$$

$$w_R = -\frac{\alpha_R A_R}{\sqrt{v^2 - v_R^2}}\left[g_R\frac{\sin\psi_\gamma}{T_\gamma^{1/4}} - 2\frac{\sin\psi_1}{T_1^{1/4}}\right] \quad (v > v_R) \tag{3.21}$$

In the above equations, we have introduced the symbols

$$v_R = \tau_R b^{-1}, \quad \alpha_R = \alpha(i\tau_R), \quad \beta_R = \beta\,(i\tau_R), \quad g_R = g\,(i\tau_R)$$

$$T_1 = (\beta_R^2 z^2 + r^2 - v_R^2 t^2)^2 + 4\beta_R^2 z^2 v_R^2 t^2, \qquad c_0 = \alpha_R\beta_R^{-1} + \gamma^2\beta_R\,\alpha_R^{-1} - g_R$$

$$T_\gamma = (\alpha_R^2 z^2 + r^2 - v_R^2 t^2)^2 + 4\alpha_R^2 z^2 v_R^2 t^2, \quad A_R = (2\rho c_0 v_R)^{-1} \tag{3.22}$$

$$\psi_1 = \frac{\pi}{4} - \frac{1}{2}\tan^{-1}\frac{\beta_R^2 z^2 + r^2 - v_R^2 t^2}{2\beta_R z v_R t}, \qquad \psi_\gamma = \frac{\pi}{4} - \frac{1}{2}\tan^{-1}\frac{\alpha_R^2 z^2 + r^2 - v^2_R t^2}{2\alpha_R z v_R t}$$

Formulas (3.19) and (3.20) define the field of the Rayleigh wave, provided the inequalities $\tau_R < |\eta_{p1}|$, $\tau_R < |\eta_{s1}|$ are satisfied, or

$$az\,(1 - a^2 v_R^2)^{-1/2} < t, \qquad bz\,(1 - b^2 v_R^2)^{-1/2} < t \tag{3.23}$$

If the first inequality in (3.23) is not satisfied, the first terms in the square brackets in (3.20) and (3.21) should be set equal to zero. If the second condition in (3.23) is not satisfied, the second terms in the square brackets should be set equal to zero.

Examination of (3.19) and (3.20) shows that q_R and w_R stop being continuous only for $T_1 = 0$ or $T_\gamma = 0$. Utilizing the explicit forms for T_1 and T_γ, we conclude that the Rayleigh waves have singularities only on the circle

$$r = v_R t, \qquad z = 0 \tag{3.24}$$

To determine the character of this singularity in (3.19) and (3.20) we must set $z = 0$ and examine q_R and w_R on the circle $r = v_R t$. For $v \neq v_R$ both components q_R and w_R have the same type of singularity. For example, when $z = 0$, we obtain

$$w_R = \pm \frac{\alpha_R A_R \tau_R^2}{\sqrt{(\pm v_R^2 \mp v^2)(\pm v_R^2 t^2 \mp r^2)}} \qquad (r \lesseqgtr v_R t,\ v \lesseqgtr v_R)$$

$$w_R = 0 \quad (r > v_R t,\ v < v_R), \qquad w_R = 0 \quad (r < v_R t,\ v > v_R) \tag{3.25}$$

It follows from (3.25) that the components in (3.19) and (3.20) have singularities of the form $r = v_R t$ and $\mathrm{Re}\,(v_R t - r)^{-1/2}$ at $\mathrm{Re}\,(r - v_R t)^{-1/2}$.

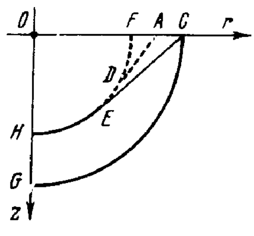

Fig. 4.

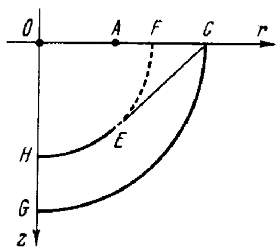

Fig. 5.

4. From Formulas (3.3), (3.6), (3.8), (3.12), (3.15), (3.16), (3.19), and (3.20), diagrams have been drawn showing the location of the fronts (Fig. 3 to 5) for the principal parts of the displacement field **u** as a function of the velocities a^{-1}, b^{-1} and v.

The results of the investigations concerning the character of singularities at these fronts are also illustrated in Fig. 3 to 5. The surface of jumps in **u** is shown in Fig. 3 to 5 by a continuous, heavy line. The front with a logarithmic singularity is shown by a heavy, broken line. The surface on which the field **u** has both jump and logarithmic singularity is shown by a dotted line. The front with a continuous field **u** is shown by a thin continuous line in the case of a discontinuous velocity **u** and by a thin dotted line in case **u** has a logarithmic singularity.

Upon examining Fig. 3 to 5, we conclude: 1) On the fronts of the $\mathbf{u}_{pv}$, $\mathbf{u}_{sv}$, $\mathbf{u}_{p0}$ and $\mathbf{u}_{s0}$ waves, the displacements **u** have jump type and logarithmic type singularities; 2) on the front of the head wave $\mathbf{u}_{s\gamma}$ the displacement field is continuous, but the velocity possesses jumps or logarithmic discontinuities; 3) on the forward fronts, the displacements have only jumps; 4) jumps and logarithmic discontinuities are observed simultaneously only on those segments of the $\mathbf{u}_{s0}$ and $\mathbf{u}_{s\gamma}$, fronts which propagate to the rear of the $\mathbf{u}_{s\gamma}$ front.

On the basis of the previously obtained relations, an explanation may also be given of the character of attenuation taking place for the principal parts of a field along a ray.

For this purpose, Expressions in (3. 3), (3. 6), (3. 8), (3. 12), (3. 15) and (3. 16) may be written, as in [6], in the form

$$I_1(r, z)\,\Phi_1(t, r, z) + I_2(r, z)\,\Phi_2(t, r, z) \tag{4.1}$$

Here, Φ_i $(i = 1, 2)$ represent integrals over the range $[k_0, \infty)$, while I_i represent the coefficients outside the integral signs.

It is easily seen that the quantities Φ_i determine the form of the waves in the frontal neighborhoods, whereas the I_i determine their intensity. Along a ray, the form remains unchanged, but the intensity decrease for $\mathbf{u}_{pv}$, $\mathbf{u}_{sv}$, $\mathbf{u}_{p0}$ and $\mathbf{u}_{s0}$ waves is proportional to t^{-1}, while for the head wave $\mathbf{u}_{s\gamma}$ it is proportional to t^{-2}. For Rayleigh waves, we will examine the attenuation with time t on the circle $r = v_R t$, $z = 0$. From (3. 25),

$$\Phi = \operatorname{Re}(r - v_R t)^{-1/2} \quad \text{or} \quad \Phi = \operatorname{Re}(v_R t - r)^{-1/2}$$

while the intensity I decreases in proportion to $t^{-1/2}$.

Let us compare waves $\mathbf{u}(v)$ produced by a moving source (1. 1) with the displacements $\mathbf{u}(0)$ and $\mathbf{u}(\infty)$ resulting from stationary sources (1. 8) and (1. 9). The displacement fields $\mathbf{u}(0)$ and $\mathbf{u}(\infty)$ as well as the investigated field $\mathbf{u}(v)$ are given by the relations (2. 3) and (2. 10), and approximate expressions for the vectors

$$\mathbf{u}_{s0},\quad \mathbf{u}_{s\gamma},\ \mathbf{u}_{p1},\ \mathbf{u}_R,\ \mathbf{u}_v = \mathbf{u}_{p\gamma} + \mathbf{u}_{sv}$$

may be obtained by the same methods as in the case investigated.

However, noting the simple relations between the sources (1. 1), (1. 8) and (1. 9), Formulas for $\mathbf{u}_{l0}(0)$, $\mathbf{u}_{l0}(\infty)$, $\mathbf{u}_{s\gamma}(0)$, $\mathbf{u}_{s\gamma}(\infty)$, $\mathbf{u}_R(0)$ and $\mathbf{u}_R(\infty)$ are easily written by utilizing the relations (3. 8), (3. 9), (3. 12), (3. 15), (3. 16), (3. 20) and (3. 21). In fact, formulas (3. 8), (3. 12), (3. 15), and (3. 20) for $v \to 0$ determine the vectors $\mathbf{u}_{p0}(0)$, $\mathbf{u}_{s0}(0)$, $\mathbf{u}_{s\gamma}(0)$ and $\mathbf{u}_R(0)$. If the right-hand sides of (3. 9), (3. 12), (3. 16) and (3. 21) are multiplied by v, and the products are subjected to a limiting procedure for $v \to \infty$, we obtain

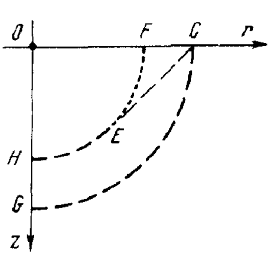

Fig. 6.

Expressions for $\mathbf{u}_{p0}(\infty)$, $\mathbf{u}_{s0}(\infty)$, and $\mathbf{u}_{s\gamma}(\infty)$ $\mathbf{u}_R(\infty)$ Equations for the remaining vectors $\mathbf{u}_{p1}(0)$, $\mathbf{u}_{p1}(\infty)$ and $\mathbf{u}_v(0)$ $(\mathbf{u}_v(\infty) \equiv 0)$ are obtainable in an elementary way, but these vectors are unnecessary. Based on studies of the vector fields $\mathbf{u}(0)$ and $\mathbf{u}(\infty)$, we may obtain the fronts, the character of singularities at the fronts (Fig. 5, $\mathbf{u}(0)$; Fig. 6, $\mathbf{u}(\infty)$) and attenuation along rays.

If the principal parts of the fields produced by the sources (1. 12) and (1. 13) are known, then we can determine the vectors $\mathbf{u}_{p0}(v)$, $\mathbf{u}_{s0}(v)$, $\mathbf{u}_{s\gamma}(v)$ and $\mathbf{u}_R(v)$ for the case of a moving source (1. 1). This problem is solved by the relations

$$\mathbf{u}_{p0}(v) = \begin{cases} (1 - a^2v^2\sin^2\varphi)^{-1/2}\,\mathbf{u}_{p0}(0) & (av\sin\varphi < 1) \\ a\sin\varphi\,(a^2v^2\sin^2\varphi - 1)^{-1/2}\,\mathbf{u}_{p0}(\infty) & (av\sin\varphi > 1) \end{cases}$$

$$\mathbf{u}_{s0}(v) = \begin{cases} (1 - b^2v^2\sin^2\psi)^{-1/2}\,\mathbf{u}_{s0}(0) & (bv\sin\psi < 1) \\ b\sin\psi\,(b^2v^2\sin^2\psi - 1)^{-1/2}\,\mathbf{u}_{s0}(\infty) & (bv\sin\psi > 1) \end{cases} \tag{4.2}$$

$$\mathbf{u}_{s\gamma}(v) = \begin{cases} (1 - a^2v^2)^{-1/2}\,\mathbf{u}_{s\gamma}(0) & (av < 1) \\ a\,(a^2v^2 - 1)^{-1/2}\,\mathbf{u}_{s\gamma}(\infty) & (av > 1) \end{cases}$$

$$\mathbf{u}_R(v) = \begin{cases} v_R\,(v_R^{\,2} - v^2)^{-1/2}\,\mathbf{u}_R(0) & (v < v_R) \\ (v^2 - v_R^{\,2})^{-1/2}\,\mathbf{u}_R(\infty) & (v > v_R) \end{cases} \tag{4.3}$$

which are based on (3. 8), (3. 9), (3. 12), (3. 15), (3. 16), (3. 20), and (3. 21). Upon examining (4. 3), we find that the head and Rayleigh waves for a moving source are equivalent, except for a factor, to the correspondingly named waves in the case of stationary sources (1. 8) and (1. 9). For the longitudinal wave $\mathbf{u}_{p0}(v)$ and the transverse wave $\mathbf{u}_{s0}(v)$ these waves are only locally equivalent to the corresponding waves $\mathbf{u}_{p0}(0)$, $u_{p0}(\infty)$, $\mathbf{u}_{s0}(0)$ and $\mathbf{u}_{s0}(\infty)$. Moreover, the wave pattern at the fronts of $\mathbf{u}_{p0}(v)$ and $\mathbf{u}_{s0}(v)$ is different in its dependence on the relationship between the angles of propagation φ and ψ and their corresponding limiting angles $\sin^{-1}(av)^{-1}$ and $\sin^{-1}(bv)^{-1}$.

5. Now let us examine the case in which the source velocity v and the propagation velocities a^{-1}, b^{-1} and v_R coincide.

If $v = a^{-1}$, the displacement field $\mathbf{u}$ is given by (1. 5) wherein (1. 6) is replaced by the following relations

$$W = \frac{\gamma}{\mu R}\,(ge^{-kz\alpha} - 2e^{-kz\beta}), \quad Q = \frac{\gamma}{\mu R\alpha}\,(ge^{-kz\alpha} - 2\alpha\beta e^{-kz\beta}) \tag{5.1}$$

Examination of the field (1. 5) and (5. 1) shows that the location of the fronts in this case is given by (3. 11), (3. 14), (3. 17) and (3. 23), while the character of the singularities at these fronts is as illustrated in Fig. 5. Approximate expressions for $\mathbf{u}_{p0}(a^{-1})$, $\mathbf{u}_{s0}(a^{-1})$ and $\mathbf{u}_R(a^{-1})$ are obtained from (3. 8), (3. 12), (3. 21), (4. 2) and (4. 3) for $v = a^{-1}$. With regard to the $\mathbf{u}_{s\gamma}(v)$, wave, its intensity as a function of v has a singularity which is related to the coincidence of the $\mathbf{u}_{s\gamma}$ and $\mathbf{u}_{sv}$ wave fronts. The wave represented by the vector $\mathbf{u}_{s\gamma}(a^{-1}) + \mathbf{u}_{sv}(a^{-1})$, has the same form in the frontal neighborhood (3. 18) as the head wave. The intensity of the wave $\mathbf{u}_{s\gamma}(a^{-1}) + \mathbf{u}_{sv}(a^{-1})$ decreases along a ray in proportion to t^{-2}. Thus, the location of the fronts, the singularities at the fronts and the wave attenuation along rays in the case of coincident velocities ($v = a^{-1}$) are the same as in the case of a fixed source ($v = 0$).

If the Eq. $v = b^{-1}$ holds, then the displacement field $\mathbf{u}$ is determined from (1. 5) and from the relations

$$W = \frac{\alpha}{\mu R}\,\beta\,(ge^{-kz\alpha} - 2e^{-kz\beta}), \qquad Q = \frac{1}{\mu R\beta}\,(ge^{-kz\alpha} - 2\alpha\beta e^{-kz\beta}) \tag{5.2}$$

As in the case $v = a^{-1}$, the vectors $\mathbf{u}_{p0}(b^{-1})$, $\mathbf{u}_{s0}(b^{-1})$, $\mathbf{u}_{s\gamma}(b^{-1})$ and $\mathbf{u}_R(b^{-1})$ will represent the principal part of the field $\mathbf{u}$: the fronts are determined from Eqs. (3. 11), (3. 14), (3. 18) and (3. 24), and the singularity types are as shown in Fig. 5. Since Eqs. (3. 8), (3. 12), (3. 15), (3. 24), (4. 2) and (4. 3) may be used for the determination of the vectors, the reduction of intensity along a ray will be the same as for $v = 0$

For $v = v_R$ a study of the waves $\mathbf{u}_{p0}(v_R)$, $\mathbf{u}_{s0}(v_R)$ and $\mathbf{u}_{s\gamma}(v_R)$ by means of relations (3. 8), (3. 12), (3. 15), (4. 2) and (4. 3) yields nothing new. Of greater interest in the case of $v = v_R$ are the $\mathbf{u}_v$ and $\mathbf{u}_R$ waves. If $v \approx v_R$, then both of these waves must be examined jointly, and the asymptotic evaluation of the integrals along the contours λ_{pv} and λ_{sv} must take into account the proximity of the poles $\pm i\tau_R$ to the branch points $\pm ibv$.

If the method of steepest descent proposed by V. A. Fok is used and the results thus obtained are subjected to a limiting procedure for $v \to v_R$, we obtain the approximate

$$q_{pv} + q_{pR} = -\frac{q_R B_p \exp(-k_0 z_p)}{Z_p^2 + R_1^2}\left[Z_p \cos\left(k_0 R_1 + \frac{\psi_p}{2}\right) - R_1 \sin\left(k_0 R_1 + \frac{\psi_p}{2}\right)\right]$$

$$q_{sv} + q_{sR} = \frac{2\alpha_R \beta_R B_s \exp(-k_0 z_s)}{Z_s^2 + R_1^2}\left[Z_s \cos\left(k_0 R_1 + \frac{\psi_s}{2}\right) - R_1 \sin\left(k_0 R_1 + \frac{\psi_s}{2}\right)\right]$$

$$w_{pv} + w_{pR} = \frac{\alpha_R g_R B_p \exp(-k_0 z_p)}{Z_p^2 + R_1^2}\left[Z_p \sin\left(k_0 R_1 + \frac{\psi_p}{2}\right) + R_1 \cos\left(k_0 R_1 + \frac{\psi_p}{2}\right)\right]$$

$$w_{sv} + w_{sR} = -\frac{2\alpha_R B_s \exp(-k_0 z_s)}{Z_s^2 + R_1^2}\left[Z_s \sin\left(k_0 R_1 + \frac{\psi_s}{2}\right) + R_1 \cos\left(k_0 R_1 + \frac{\psi_s}{2}\right)\right] \tag{5.3}$$

wherein

$$B_l = \frac{\sqrt{A_l}}{4c_0\rho v_R \sqrt{r v_R}}, \qquad \psi_p = \tan^{-1} \frac{a^2 v_R z}{\alpha_R t}, \qquad \psi_s = \tan^{-1} \frac{b^2 v_R z}{\beta_R t}$$

$$A_p = \left(t^2 + \frac{a^4 v_R^2 z^2}{\alpha_R^2}\right)^{1/2}, \quad A_S = \left(t^2 + \frac{b^4 v_R^2 z^2}{\beta_R^2}\right)^{1/2}, \quad R_1 = r - v_R t,\ Z_p = \alpha_R z,\ Z_s = \beta_R z$$

From (5.3), we find that $\mathbf{u}_R + \mathbf{u}_v$ ceases to be continuous only on the circle (3.24). Analysis of the Expressions (5.3) in the neighborhood $r = v_R t$, $z = 0$ shows that w has a singularity of the type $(r - v_R t)^{-1}$ in the $z = 0$ plane, while the component q goes to infinity like z^{-1}, on the surface $r = v_R t$. As t increases, the intensity of the wave represented by $\mathbf{u}_{lv} + \mathbf{u}_{lR}$, remains a constant on the circle (3.24).

The absence of attenuation and the change of form at the front of the wave $\mathbf{u}_v(v_R) + \mathbf{u}_R(v_R)$ are connected with the resonance phenomenon for Rayleigh waves, which is known from previous studies (for example, [1 to 3]). This resonance takes place when the source velocity coincides with the Rayleigh velocity. The variation in intensity for large t is different from that given in [3]. This, however, does not imply any contradiction. Indeed, for large t the relation between the Rayleigh wave intensity in (5.3) and the source in (1.1) is found to be proportional to t. This is identical with the relation obtained by using the results of [3].

Investigation of the cases in which the source velocity coincides with the propagation velocities thus shows that resonance will occur only for the Rayleigh surface wave when $v = v_R$. This is not surprising since the source (1.1) which is being investigated is of the surface type. To obtain resonance for a volume wave $\mathbf{u}_{p0}$ the source must be applied over an expanding sphere or hemisphere and the pertinent velocity equation is $v = a^{-1}$.

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